Nonparametric Risk Attribution for Factor Models of Portfolios

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Outline

The problem
Risk decomposition

- Factor models are used to forecast returns of a portfolio with known positions in underlying assets
- These models are used to predict **risk**
  - Volatility, value-at-risk (VaR), and expected tail loss (ETL)
- Clients would often like to identify sources of risk in their portfolio
  - Goal: divide up the overall risk among the factors

The problem
Risk decomposition

- A function is **homogeneous of order one** if
- VaR and ETL have this property.
  - VaR: the upper alpha **quantile** of the loss distribution

$$VaR_\alpha(R) = -\inf_x\{\mathbb{P}_F(R < x) \geq \alpha\}$$

  - ETL: the **conditional expectation** of the upper alpha tail of the loss distribution

$$ETL_\alpha(R) = \mathbb{E}(R \mid R \leq -VaR_\alpha)$$

- **Euler's formula**
  - If a portfolio is comprised of K assets with known weights β_k, and the risk measure is homogeneous of order one, then

$$r_P = \sum_{k=1}^{K} \beta_k \frac{\partial r_P}{\partial \beta_k}$$

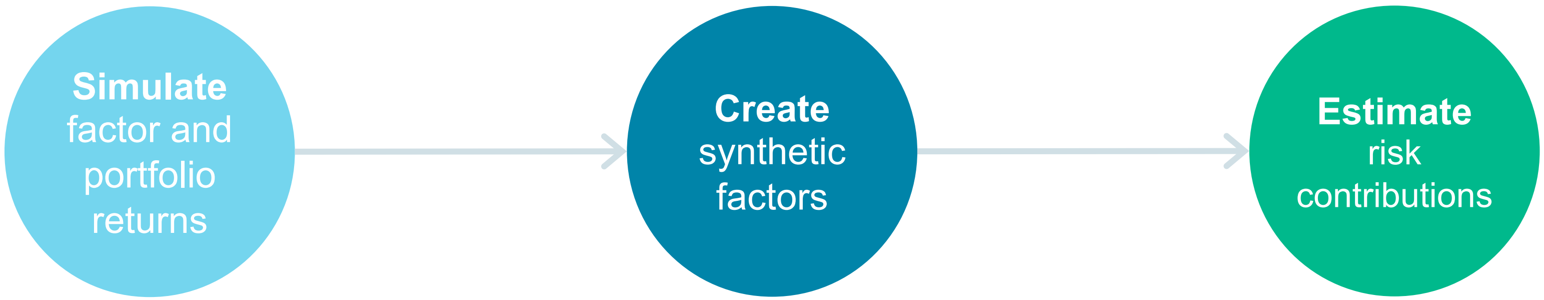
The problem
Risk decomposition

- What's wrong with using Euler's formula?

$$r_P = \sum_{k=1}^{K} \beta_k \frac{\partial r_P}{\partial \beta_k}$$

- If the portfolio is a nonlinear function of factors, then this additive decomposition does **not** hold.
  - Nonlinear instruments (e.g. bonds, derivatives)
  - Simple vs. log returns
- An asset's marginal contribution to risk is **not** its standalone risk
  - Incremental contribution to risk is $\frac{\partial r_P}{\partial \beta_k}$
  - We have to estimate derivatives

Solution



The portfolio is nonlinear in the factors. Previous results for partitioning portfolio risk do not apply.

The portfolio is linear in the synthetic factors. We estimate them using the generalized additive model framework.

The additive decomposition applies. The portfolio risk is partitioned amongst nonlinear transformations of the original factors.

All estimation can be done **nonparametrically**:
we don’t need to make any distributional assumptions.

Additive models

An algorithm for creating “synthetic factors” which are linearly related to portfolio returns.

Additive models

Problem

- Factors influence the underlying assets which make up the portfolio.
- A linear relationship between factor and portfolio returns would allow us to use the Euler decomposition to attribute risk to factors.
- If the portfolio were made up of solely linear instruments (e.g. stocks), then we could estimate the factor contributions using linear regression

$$R_P = \hat{\beta}' F + \hat{\varepsilon}$$

- But, factors influence portfolio returns in a **nonlinear** way when other instruments are present so this may be a bad approximation.

Additive models

Solution

- If factors are **independent**, then there exists an additive decomposition called the **Hajek projection**:

$$\hat{R}_P = \mathbb{E}(R_P) + \sum_{k=1}^{K} (\mathbb{E}(R_P \mid F_k) - \mathbb{E}(R_P)) = \mathbb{E}(R_P) + \sum_{k=1}^{K} g_k(F_k)$$

  – $g_k(F_k)$ are measurable functions with finite second moments

  – They can take any form and may vary for each factor.

- **Idea:** Create synthetic factors $\tilde{F}_k = \hat{g}_k(F_k)$
- Then the portfolio return is a sum of synthetic factors

$$R_P = \overline{R_P} + \sum_{k=1}^{K} \tilde{F}_k + \varepsilon$$

Additive models

Estimation

- Iterate through factors and estimate in sequence, since they are independent
- Estimate relationship between the portfolio returns and the residuals of predicted returns using all **other** factors

Algorithm 1 Backfitting algorithm for GAMs

$\hat{\alpha} \leftarrow \frac{1}{N}\sum_{i=1}^{N} y_i, \hat{g}_k \leftarrow 0, \forall k$
while $\hat{g}_k$ have not converged **do**
  for $k = 1, \ldots, K$ **do**
    $\hat{g}_k \leftarrow \text{Smooth}\left(y_i - \hat{\alpha} - \sum_{j \neq k} \hat{g}_k(F_{ij})\right)$
    $\hat{g}_k \leftarrow \hat{g}_k - \frac{1}{N}\sum_{i=1}^{N} \hat{g}_k(F_{ik})$
  end for
end while

- Smooth can be any regression method. We’ll discuss specific methods later.

Euler’s formula

A theorem that allows us to decompose the risk of a portfolio into a sum of risk contributions

Euler's formula

Solution

- Euler's formula: $$r_P = \sum_{k=1}^{K} \beta_k \frac{\partial r_P}{\partial \beta_k}$$

- How do we calculate these derivatives?

- Hallerbach (1999), Tasche (1999), and Gourieroux et al. (2000), and others show that these partial derivatives can also be expressed as expected values:

$$\frac{\partial VaR_\alpha(R)}{\partial \beta_k} = \mathbb{E}(F_k \mid -R_P = VaR)$$

$$\frac{\partial ETL_\alpha(R)}{\partial \beta_k} = \mathbb{E}(F_k \mid -R_P \geq VaR)$$

- The problem of taking derivatives is just a regression problem!

Nonparametric regression

Moving beyond least squares to find a “best fit” to the data.

Nonparametric regression

Problem

- Linear regression is too rigid. Not all problems are linear!
- **Idea:** modify the least squares optimization problem to give more robust solutions
- We use these methods in both estimation steps:
  - The smoother regression in creating synthetic factors
  - The expected value in estimating the marginal risk contributions

Nonparametric regression

Kernel methods

- **Idea:** do a separate regression at each input value, giving more weight to nearby observations
- Weighting is done using a **kernel**, a function made up of a distance between inputs and bandwidth h

$$K_h(x, y) = K\left(\frac{x-y}{h}\right)$$

  - If x and y are far apart, the weight is small
  - Small values of h lead to more smoothing – near and far points have similar weights

- Kernel regression is a weighted linear regression problem at each x

$$\hat{f}^{LL}(x) = \arg \min_{\alpha(x),\beta(x)} \sum_{b=1}^{B} K_h(x, x_b)(\alpha(x) + \beta(x)x_b - y_b)^2$$

- The solution is "locally linear"

Nonparametric regression

Smoothing splines

- **Idea:** fit a cubic spline to the data, but penalize the roughness so it isn't "too wiggly"
  - **Cubic splines** interpolate the data by putting cubic polynomials between every pair of points
  - The second derivative of the function indicates roughness. Small = smooth
- The loss function trades off fit with roughness:

$$\hat{S} = \arg\min_{S} \sum_{b=1}^{B} (y_b - S(x_b))^2 + \lambda \int S''(x)^2 dx$$

- Solution can be written in closed form, so it is fast to compute

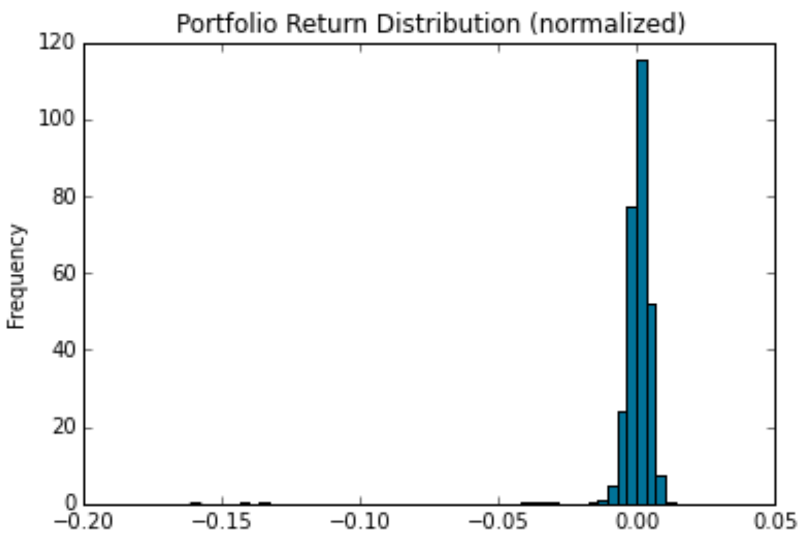
Example

Evaluating the method on simulated data from GX Labs.

Example

Portfolio

- Standard December 2009 proxy 50-50 portfolio
- 30 factors
  - First 20 are equity factors, with the 20th being most important
  - Last 10 are a mix of fixed-income and equity factors
- 10,000 simulation paths



Example

Methods

- Step 1: create synthetic factors using smoothing splines OR use raw factors
- Step 2: estimate risk contributions using smoothing splines, OLS, or formulas derived from the Gaussian distribution
- Some comments:
  - All splines were fit using default tuning parameters
  - Local linear regression was order of magnitudes slower
  - Estimated VaR and ETL at alpha=0.05
- Measure the following
  - Error: sum of risk contributions divided by estimated portfolio risk
  - Time: seconds to run the two steps for VaR and ETL

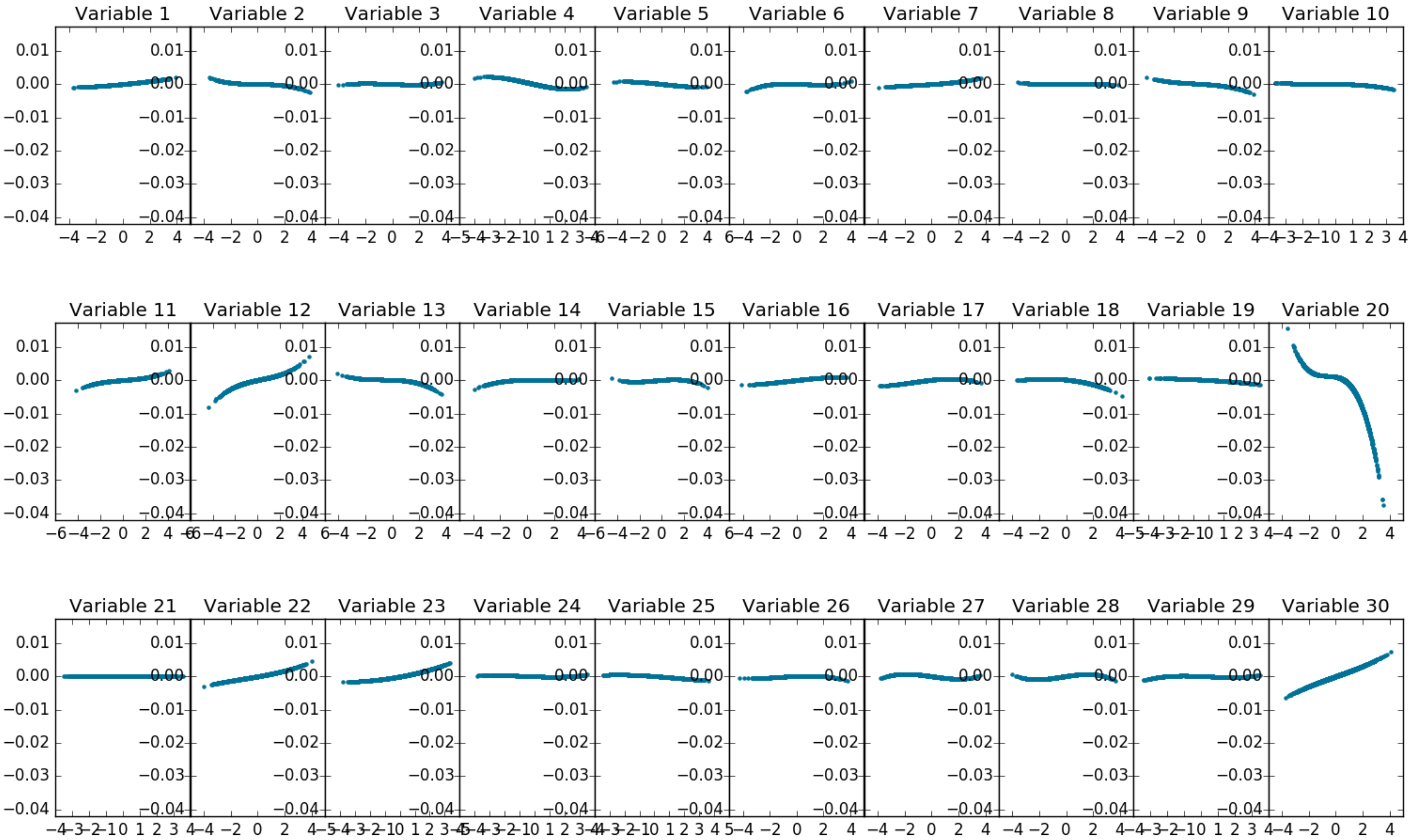
Example

Comparison of performance

Method	VaR Ratio	ETL Ratio	Time (s)
Synthetic factors + smoothing splines	0.810	0.356	13.475
Synthetic factors + OLS	0.921	0.221	14.830
Synthetic factors + Gaussian formulas	0.219	0.219	14.119
Raw factors + OLS	0.002	0.168	0.055
Raw factors + Gaussian formulas	0.168	1.002	0.047

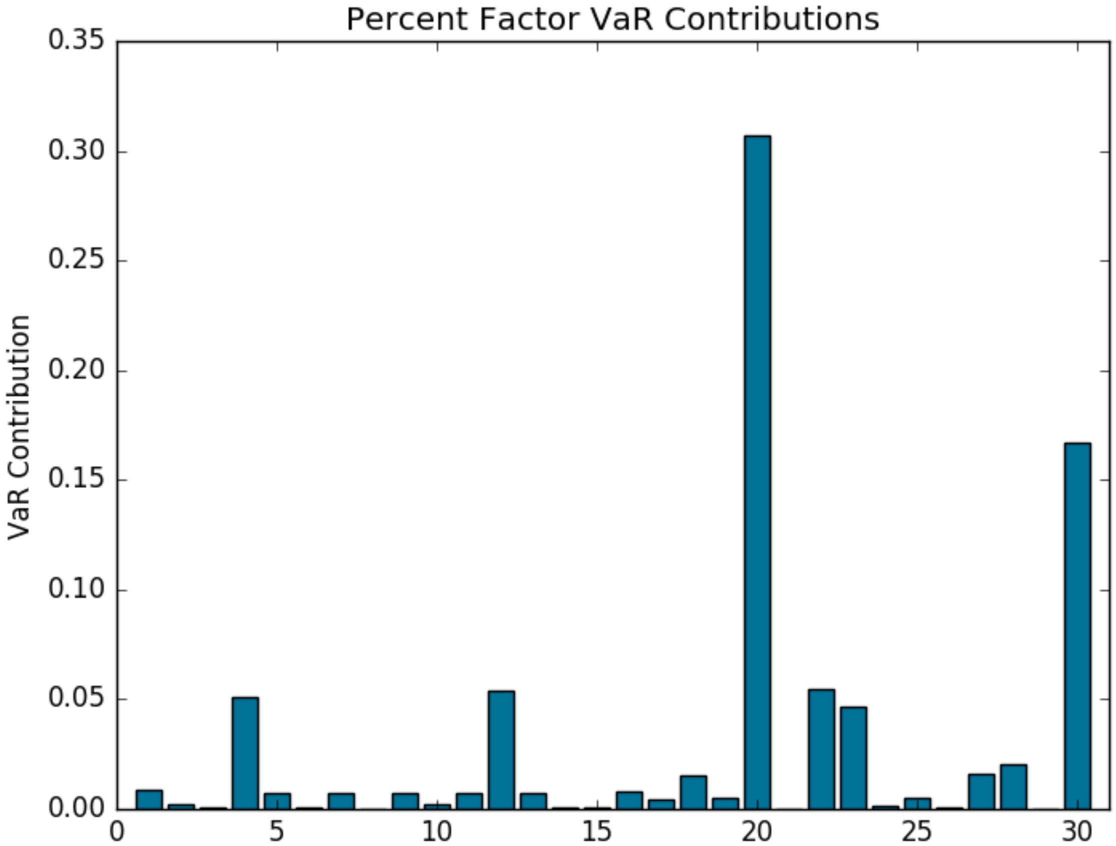
Example

Synthetic Factors



Example

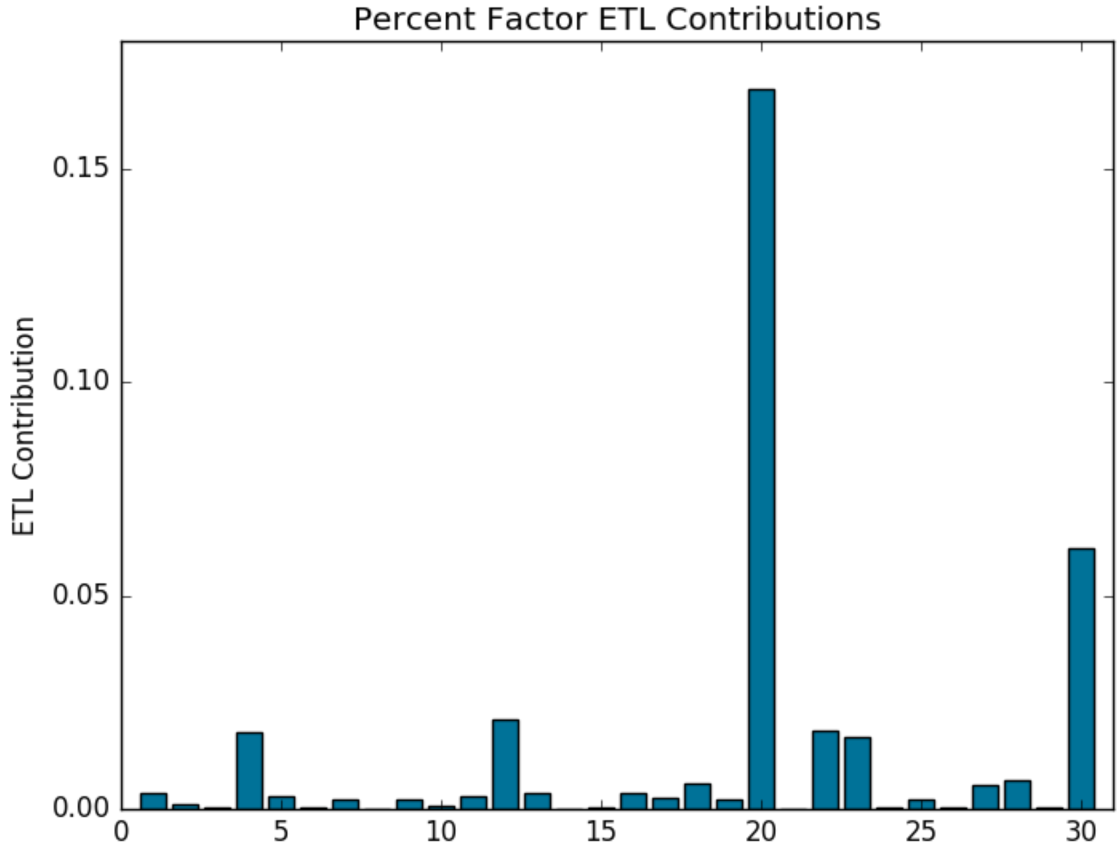
Portfolio VaR



Estimated risk contributions from smoothing splines + smoothing splines

Example

Portfolio ETL



Estimated risk contributions from smoothing splines + smoothing splines

Discussion

- Using smoothing splines to explain the relationship between factors and portfolio is computationally fast and more accurate than parametric methods that assume linearity or Gaussianity.
  - Results make sense: what we believe a priori are the most important factors have the highest contribution to risk
- Interpretability is difficult
  - Risk is attributed to synthetic factors, not raw factors
  - The relationship between factors and synthetic factors may not be monotonic, invertible, etc.
- Methods assume factors are independent
  - Instead of using the additive decomposition, one may use a **Hoeffding decomposition** which includes cross-terms. See Rosen and Saunders (2010).